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Contravariantly finite subcategories closed under predecessors

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ABSTRACT

Let A be an Artin algebra and $\operatorname{mod} A$ be the category of finitely generated right A-modules. We prove that an additive full subcategory $\mathcal{C}$ of $\operatorname{mod} A$ closed under predecessors is contravariantly finite if and only if its right Ext-orthogonal is covariantly finite, or if and only if the Ext-injectives in $\mathcal{C}$ define a cotilting module (over the support algebra of $\mathcal{C}$) or, equivalently, if and only if $\mathcal{C}$ is the support of the representable functors given by the Ext-injectives.

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Let A be an Artin algebra. We are interested in studying the representation theory of A, thus the category $\operatorname{mod} A$ of the finitely generated A-modules. For this purpose, we fix a full subcategory $\operatorname{ind} A$ of $\operatorname{mod} A$ having as objects exactly one representative of each isomorphism class of indecomposable modules. In [19], Happel, Reiten and Smalø have defined the left part $\mathcal{L}_A$ of $\operatorname{mod} A$ to be the full subcategory of $\operatorname{ind} A$ with objects those modules whose predecessors have projective dimension at most one. The right part $\mathcal{R}_A$ is defined dually. These classes, whose definitions suggest the interplay between homological properties of an algebra and representation theoretic ones, were heavily investigated and applied (see, for instance [4,6,9] and the survey [8]). In particular, it was shown that the left part of an arbitrary Artin algebra closely resembles that of a tilted algebra.

In the present paper, following a line of ideas already implicit in [4], we consider, instead of $\mathcal{L}_A$, a full subcategory $\mathcal{C}$ of $\operatorname{ind} A$ which is closed under predecessors and we give criteria for the additive subcategory $\operatorname{add}\mathcal{C}$ of $\operatorname{mod} A$ generated by $\mathcal{C}$ to be contravariantly finite (in the sense of [15]). In this more general setting, the techniques employed for the class $\mathcal{L}_A$ fail. Instead, our main tool will be the fundamental result of Auslander and Reiten linking cotilting modules (of arbitrary finite injective

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dimension) with contravariantly finite resolving subcategories [13]. As other tools, already considered in [4,9], we use firstly the properties of the Ext-injective modules in $\mathcal{C}$ (whose direct sum is denoted by E) and secondly, the support algebra ${}_{\mathcal{C}}A$ of the subcategory $\mathcal{C}$. In order to state our main result, we need more notation: following [13], we denote by $\mathcal{C}^{\perp}$ the full subcategory of $\operatorname{mod} A$ consisting of all the modules M such that $\operatorname{Ext}^i_A(-, M)|_{\mathcal{C}} = 0$ for all $i > 0$, and, following [6], we denote by $\operatorname{Supp}(-, E)$ the full subcategory consisting of all the modules M such that $\operatorname{Hom}_A(M, E) \neq 0$. Our first theorem is the following.

Theorem A. *Let $\mathcal{C}$ be a full subcategory of* $\operatorname{ind} A$*, closed under predecessors. The following conditions are equivalent*:

(a) $\operatorname{add}\mathcal{C}$ *is contravariantly finite.*
(b) $\mathcal{C}^{\perp}$ *is covariantly finite.*
(c) *E is a cotilting ${}_{\mathcal{C}}A$-module.*
(d) $\operatorname{add}\mathcal{C} = \operatorname{Supp}(-, E)$.
(e) *Any morphism $f : L \longrightarrow M$ with $L \in \mathcal{C}$ and M indecomposable not in $\mathcal{C}$ factors through* $\operatorname{add} E$.

Clearly, in general, the cotilting ${}_{\mathcal{C}}A$-module E is not tilting, because it may have infinite projective dimension. However, the following finiteness assumption allows to generalise the main results of [4,6,9], see Remarks in 4.4. Let $\operatorname{pgd}\mathcal{C}$ denote the supremum of the projective dimensions of the modules in $\mathcal{C}$, and F denote the direct sum of all the indecomposable projective A-modules not lying in $\mathcal{C}$. Our second theorem is the following.

Theorem B. *Let $\mathcal{C}$ be a full subcategory of* $\operatorname{ind} A$ *closed under predecessors and such that* $\operatorname{pgd}\mathcal{C} < \infty$. *The following statements are equivalent*:

(a) $\operatorname{add}\mathcal{C}$ *is contravariantly finite.*
(b) *E is a tilting ${}_{\mathcal{C}}A$-module.*
(c) *$T = E \oplus F$ is a tilting A-module.*

Moreover, in this case, $\mathcal{C}^{\perp} = T^{\perp} = E^{\perp}$, and $\mathcal{C}$ consists of all the predecessors of E in $\operatorname{ind} A$.

As an application of these theorems, we generalise [10, (2.1)] which characterises tilted algebras as being those algebras having a convex tilting module of projective dimension at most one. Namely, we prove the following theorem.

Theorem C. *An algebra is tilted if and only if it has a convex tilting* (*or cotilting*) *module.*

In a forthcoming work with E.R. Alvares and M.I. Peña, we further apply our results to the study of trisections (see [1]).

The paper is organised as follows. The first section contains the needed notation and preliminaries on tilting and cotilting modules. In the second section, we consider the particular case when we deal with a resolving subcategory. We consider the existence of tilting modules in Section 3 and prove our main theorems in Section 4. Finally, Section 5 contains the application to tilted algebras.

Clearly, the dual results, for the covariant finiteness of full subcategories of $\operatorname{ind} A$ closed under successors, hold as well. For the sake of brevity, we refrain from stating them, leaving the primal-dual translation to the reader.

1. Preliminaries on tilting modules

1.1. Notation

Throughout this paper, all our algebras are basic and connected Artin algebras. For an algebra A, we denote by $\operatorname{mod} A$ the category of finitely generated right A-modules and by $\operatorname{ind} A$ a full subcategory of $\operatorname{mod} A$ consisting of exactly one representative from each isomorphism class of indecomposable modules. When we speak about a module (or an indecomposable module), we always mean implicitly that it belongs to $\operatorname{mod} A$ (or to $\operatorname{ind} A$, respectively). Also, all subcategories of $\operatorname{mod} A$ are full and so are identified with their object classes.

A subcategory $\mathcal{C}$ of $\operatorname{ind} A$ is called *finite* if it has only finitely many objects. We sometimes write $M \in \mathcal{C}$ to express that M is an object in $\mathcal{C}$. We denote by $\operatorname{add}\mathcal{C}$ the subcategory of $\operatorname{mod} A$ with objects the finite direct sums of summands of modules in $\mathcal{C}$ and, if M is a module, we abbreviate $\operatorname{add}\{M\}$ as $\operatorname{add} M$. We denote the projective (or injective) dimension of a module M as $\operatorname{pd} M$ (or $\operatorname{id} M$, respectively). The global dimension of A is denoted by $\operatorname{gl.dim} A$. If $\mathcal{C}$ is a subcategory of $\operatorname{ind} A$, we define its *projective global dimension* $\operatorname{pgd}(\mathcal{C})$ (or its *injective global dimension* $\operatorname{igd}(\mathcal{C})$) to be the supremum of the projective (or the injective, respectively) dimensions of the modules lying in $\mathcal{C}$. For a module M, the support $\operatorname{Supp}(M,-)$ (or $\operatorname{Supp}(-,M)$) of the functor $\operatorname{Hom}_A(M,-)$ (or $\operatorname{Hom}_A(-,M)$) is the subcategory of $\operatorname{ind} A$ consisting of all modules X such that $\operatorname{Hom}_A(M,X) \neq 0$ (or $\operatorname{Hom}_A(X,M) \neq 0$, respectively). We denote by $\operatorname{Gen} M$ (or $\operatorname{Cogen} M$) the subcategory of $\operatorname{mod} A$ having as objects all modules generated (or cogenerated, respectively) by M.

For an algebra A, we denote by $\Gamma(\operatorname{mod} A)$ its Auslander–Reiten quiver and by $\tau_A = \mathrm{DTr}$, $\tau_A^{-1} = \mathrm{TrD}$ its Auslander–Reiten translations. For further definitions and facts needed on $\operatorname{mod} A$ or $\Gamma(\operatorname{mod} A)$, we refer the reader to [12,14].

1.2. Tilting modules

An A-module T is called *auto-orthogonal* if $\operatorname{Ext}_A^i(T,T) = 0$ for all $i > 0$. An A-module T is called a *tilting module* if it is auto-orthogonal, of finite projective dimension and there is an exact sequence

$$0 \longrightarrow A_A \longrightarrow T_0 \longrightarrow T_1 \longrightarrow \cdots \longrightarrow T_r \longrightarrow 0$$

with $T_i \in \operatorname{add} T$ for all i. The dual notion is that of cotilting module, namely, an A-module T is a *cotilting module* if it is auto-orthogonal, of finite injective dimension and there is an exact sequence

$$0 \longrightarrow T_r \longrightarrow \cdots \longrightarrow T_1 \longrightarrow T_0 \longrightarrow \mathrm{D}A_A \longrightarrow 0$$

with $T_i \in \operatorname{add} T$ for all i.

Given a module T, we define its *right orthogonal* $T^{\perp}$ to be the full subcategory of $\operatorname{mod} A$ with object class

$$T^{\perp} = \{X \in \operatorname{mod} A\colon \operatorname{Ext}_A^i(T,X) = 0, \text{ for all } i > 0\}.$$

We define similarly its *left orthogonal* ${}^{\perp}T$ by

$${}^{\perp}T = \{X \in \operatorname{mod} A\colon \operatorname{Ext}_A^i(X,T) = 0, \text{ for all } i > 0\}.$$

We need the following result of D. Happel [17, Section 3].

Theorem. *Let T be an auto-orthogonal module of finite projective dimension. Then T is a tilting module if and only if $T^{\perp} \subset \operatorname{Gen} T$.*

1.3. Covariant and contravariant finiteness

Let $\mathcal{X}$ be an additive subcategory of $\operatorname{mod} A$. For an A-module M, a *right $\mathcal{X}$-approximation of M* is a morphism $f_M : X_M \longrightarrow M$ with $X_M \in \mathcal{X}$ such that any morphism $f : X \longrightarrow M$ with $X \in \mathcal{X}$ factors through f_M. The morphism f_M is also called *right minimal* if $f_M \circ h = f_M$ for a morphism h implies that h is an automorphism. The subcategory $\mathcal{X}$ is called *contravariantly finite* if any A-module has a right $\mathcal{X}$-approximation. We define dually *left $\mathcal{X}$-approximations, left minimal $\mathcal{X}$-approximations* and *covariantly finite subcategories*. Finally, $\mathcal{X}$ is called *functorially finite* if it is both contravariantly and covariantly finite. Observe that any subcategory having only finitely many isomorphism classes of indecomposables is functorially finite (see [15]).

The subcategory $\mathcal{X}$ is called *coresolving* if it is closed under extensions, under cokernels of monomorphisms and contains all the injective A-modules. The dual notion is that of a resolving subcategory.

We define $\check{\mathcal{X}}$ to be the full subcategory of $\operatorname{mod} A$ whose objects are all the $M \in \operatorname{mod} A$ for which there is an exact sequence

$$0 \longrightarrow M \longrightarrow X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_m \longrightarrow 0$$

with $X_i \in \mathcal{X}$ for all i. Dually, $\widehat{\mathcal{X}}$ is the full subcategory whose objects are all the $M \in \operatorname{mod} A$ for which there is an exact sequence

$$0 \longrightarrow X'_n \longrightarrow \cdots \longrightarrow X'_1 \longrightarrow X'_0 \longrightarrow M \longrightarrow 0$$

with $X'_j \in \mathcal{X}$ for all j. Finally, a module T is called *multiplicity-free* if $T = \bigoplus_{k=1}^{s} T_k$ with all T_k indecomposable implies T_k not isomorphic to T_l, for $k \neq l$. We need the following fundamental result of Auslander and Reiten [13, (5.5)].

Theorem. *Let T be an auto-orthogonal module. Then $T \mapsto T^{\perp}$ gives a bijection between the isomorphism classes of multiplicity-free tilting modules and covariantly finite coresolving subcategories $\mathcal{X}$ such that $\check{\mathcal{X}} =$* $\operatorname{mod} A$.

If $\operatorname{gl.dim} A < \infty$, then $\check{\mathcal{X}} = \operatorname{mod} A$ for any coresolving subcategory $\mathcal{X}$, so the statement holds without this condition.

1.4. We need the following statement, whose proof follows the same line as [2], where infinitely generated modules over a ring are considered.

Lemma. *Let T be an auto-orthogonal module of finite projective dimension. Then T is tilting if and only if, for each $M \in T^{\perp}$, there exists a right minimal* $\operatorname{add} T$*-approximation $f_0 : T_0 \longrightarrow M$ which is an epimorphism such that* $\operatorname{Ker} f_0 \in T^{\perp}$.

Proof. The sufficiency follows at once from 1.2 since the stated condition says that any $M \in T^{\perp}$ belongs to $\operatorname{Gen} T$. We thus prove the necessity. Assume T is tilting, let $f_0 : T_0 \longrightarrow M$ be a right minimal $\operatorname{add} T$-approximation and $K_0 = \operatorname{Ker} f_0$. Because $M \in T^{\perp}$ and $T^{\perp} \subset \operatorname{Gen} T$, there exist $d > 0$ and an epimorphism $p : T^d \longrightarrow M$. Since p factors through f_0, the latter is also an epimorphism, so we have a short exact sequence

$$0 \longrightarrow K_0 \longrightarrow T_0 \xrightarrow{f_0} M \longrightarrow 0.$$

We claim that $K_0 \in T^{\perp}$. Applying $\operatorname{Hom}_A(T, -)$ yields an exact sequence

$$\begin{aligned} 0 &\longrightarrow \operatorname{Hom}_A(T, K_0) \longrightarrow \operatorname{Hom}_A(T, T_0) \longrightarrow \operatorname{Hom}_A(T, M) \\ &\longrightarrow \operatorname{Ext}^1_A(T, K_0) \longrightarrow \operatorname{Ext}^1_A(T, T_0) \longrightarrow \operatorname{Ext}^1_A(T, M) \longrightarrow \cdots \\ &\longrightarrow \operatorname{Ext}^{i-1}_A(T, M) \longrightarrow \operatorname{Ext}^i_A(T, K_0) \longrightarrow \operatorname{Ext}^i_A(T, T_0) \longrightarrow \cdots. \end{aligned}$$

Since f_0 is an $\operatorname{add} T$-approximation and T is auto-orthogonal, we have $\operatorname{Ext}^1_A(T, K_0) = 0$. The same auto-orthogonality and the hypothesis that $M \in T^{\perp}$ imply $\operatorname{Ext}^i_A(T, K_0) = 0$, for all $i \geqslant 2$. Thus $K_0 \in T^{\perp}$. $\square$

1.5. We recall that, if $\mathcal{X}$ is an additive subcategory of $\operatorname{mod} A$, closed under extensions, then a module $M \in \mathcal{X}$ is called *Ext-projective* (or *Ext-injective*) in $\mathcal{X}$ if $\operatorname{Ext}^1_A(M, -)|_{\mathcal{X}} = 0$ (or $\operatorname{Ext}^1_A(-, M)|_{\mathcal{X}} = 0$, respectively), see [16]. It is shown in [16, (3.3), (3.7)] that, if $\mathcal{X}$ is a torsion (or torsion-free) class, then an indecomposable module $M \in \mathcal{X}$ is Ext-projective if and only if $\tau_A M$ is torsion-free (or, $M \in \mathcal{X}$ is Ext-injective if and only if $\tau_A^{-1} M$ is torsion).

Corollary. *Let T be a tilting module. Then $X \in \operatorname{add} T$ if and only if X is Ext-projective in $T^{\perp}$.*

Proof. Clearly, if $X \in \operatorname{add} T$, then X is Ext-projective in $T^{\perp}$. Conversely, assume X is Ext-projective in $T^{\perp}$. Consider the exact sequence

$$0 \longrightarrow K_0 \longrightarrow T_0 \xrightarrow{f_0} X \longrightarrow 0$$

as in 1.4. Since $K_0 \in T^{\perp}$, the Ext-projectivity of X implies that it splits. Hence $X \in \operatorname{add} T$. $\square$

1.6.

Lemma. *Let A be an algebra such that $\operatorname{pd} DA < \infty$ and T be a tilting module of finite injective dimension. Then T is a cotilting module of finite projective dimension.*

Proof. Since T is a tilting module, then T is auto-orthogonal and $\operatorname{pd} T < \infty$. Because, clearly, $DA \in T^{\perp}$, we have a short exact sequence as in 1.4

$$0 \longrightarrow K_0 \longrightarrow T_0 \xrightarrow{f_0} DA \longrightarrow 0$$

with $f_0 : T_0 \longrightarrow DA$ a right minimal $\operatorname{add} T$-approximation and $K_0 \in T^{\perp}$. Inductively, we construct an exact sequence

$$\cdots \longrightarrow T_2 \xrightarrow{f_2} T_1 \xrightarrow{f_1} T_0 \xrightarrow{f_0} DA \longrightarrow 0$$

with $T_i \in \operatorname{add} T$ for all i, and $K_i = \operatorname{Ker} f_i \in T^{\perp}$.

In order to show that T is a cotilting A-module, it suffices to prove that the above sequence terminates. Since $d = \operatorname{pd} DA < \infty$, then $\operatorname{Ext}^{d+1}_A(DA, K_d) = 0$. The short exact sequences

$$0 \longrightarrow K_j \longrightarrow T_j \longrightarrow K_{j-1} \longrightarrow 0$$

yield, for each $i > 0$, an isomorphism

$$\operatorname{Ext}^i_A(K_j, K_d) \cong \operatorname{Ext}^{i+1}_A(K_{j-1}, K_d).$$

Applying repeatedly this formula yields

$$\operatorname{Ext}^1_A(K_{d-1}, K_d) \cong \cdots \cong \operatorname{Ext}^{d+1}_A(DA, K_d) = 0.$$

This completes the proof. $\square$

1.7. We recall from [13,18], that an algebra A is *Gorenstein* if $\operatorname{pd} DA < \infty$ and $\operatorname{id} A < \infty$. Clearly, if $\operatorname{gl.dim} A < \infty$, then A is Gorenstein.

Corollary. *Let A be a Gorenstein algebra. Then T is a tilting module of finite injective dimension if and only if T is a cotilting module of finite projective dimension.*

2. The resolving case

2.1. Paths

Given $M, N \in \operatorname{ind} A$, a *path* from M to N, denoted by $M \rightsquigarrow N$, is a sequence of non-zero morphisms

$$M = X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \longrightarrow X_{t-1} \xrightarrow{f_t} X_t = N \tag{$*$}$$

($t \geqslant 1$) where $X_i \in \operatorname{ind} A$ for all i. We then say that M is a *predecessor* of N and N is a *successor* of M. The length of the path ($*$) is t. A path from M to M involving at least one non-isomorphism is a *cycle*. A module $M \in \operatorname{ind} A$ which lies on no cycle is called *directed*. If each f_i in ($*$) is irreducible, we say that ($*$) is a path of irreducible morphisms, or a path in $\Gamma(\operatorname{mod} A)$. A path ($*$) of irreducible morphisms is called *sectional* if $\tau_A X_{i+1} \neq X_{i-1}$ for all i with $0 < i < t$. A *refinement* of ($*$) is a path

$$M = X'_0 \longrightarrow X'_1 \longrightarrow \cdots \longrightarrow X'_{s-1} \longrightarrow X'_s = N$$

in $\operatorname{ind} A$ such that there is an order-preserving injection

$$\sigma : \{1, 2, \ldots, t-1\} \longrightarrow \{1, 2, \ldots, s-1\}$$

satisfying $X_i = X'_{\sigma(i)}$ for all i with $0 < i < t$.

Lemma. *Let $X, Y \in \operatorname{ind} A$. If, for some $i \geqslant 1$, we have $\operatorname{Ext}^i_A(X, Y) \neq 0$, then there exists a path in $\operatorname{ind} A$ from Y to X of length $i+1$.*

Proof. By induction on i. This is clear if $i = 1$. Assume $i > 1$ and consider the short exact sequence

$$0 \longrightarrow K \longrightarrow P \xrightarrow{p} X \longrightarrow 0$$

where p is a projective cover. Then

$$\operatorname{Ext}^{i-1}_A(K, Y) \cong \operatorname{Ext}^i_A(X, Y) \neq 0.$$

Hence there exists an indecomposable summand Z of K such that $\operatorname{Ext}^{i-1}_A(Z, Y) \neq 0$. By the induction hypothesis, there exists a path $Y \rightsquigarrow Z$ of length i. Since the short exact sequence above does not split, there exists a summand P_0 of P such that $\operatorname{Hom}_A(Z, P_0) \neq 0$. By the construction of a projective cover, we also have $\operatorname{Hom}_A(P_0, X) \neq 0$. This yields the required path $Y \rightsquigarrow Z \longrightarrow P_0 \longrightarrow X$. $\square$

2.2. A full subcategory $\mathcal{C}$ of $\operatorname{ind} A$ is called *closed under predecessors* if, whenever $M \rightsquigarrow N$ is a path in $\operatorname{ind} A$ with $N \in \mathcal{C}$, then $M \in \mathcal{C}$. We have the following easy observation.

Remark. A full subcategory $\mathcal{C}$ of $\operatorname{ind} A$ is closed under predecessors if and only if $\operatorname{add} \mathcal{C}$ is the torsion-free class of a split torsion pair.

We define dually subcategories *closed under successors* which generate torsion classes of split torsion pairs. Clearly, a full subcategory $\mathcal{C}$ of $\operatorname{ind} A$ is closed under predecessors if and only if its complement $\mathcal{C}^c = \operatorname{ind} A \setminus \mathcal{C}$ is closed under successors.

Important examples are the left and the right parts of $\operatorname{mod} A$ introduced in [19]. The *left part* of $\operatorname{mod} A$ is the full subcategory of $\operatorname{ind} A$ defined by

$$\mathcal{L}_A = \{M \in \operatorname{ind} A\colon \ \operatorname{pd} L \leqslant 1 \text{ for any predecessor } L \text{ of } M\}.$$

Clearly, $\mathcal{L}_A$ is closed under predecessors. We refer to [8,4] for characterisations of this class. The dual concept is that of the *right part* $\mathcal{R}_A$ of $\operatorname{mod} A$.

Let $\mathcal{C}$ be a full subcategory of $\operatorname{ind} A$, closed under predecessors. Then, by [4, (5.3)], the full subcategory $\mathcal{E}$ of $\operatorname{ind} A$ consisting of the indecomposable Ext-injectives in $\mathcal{C}$ is finite (that is, it contains only finitely many isomorphism classes of indecomposable objects). We set $E = \bigoplus_{X \in \mathcal{E}} X$ and denote by F the direct sum of a complete set of representatives of the isomorphism classes of the indecomposable projective A-modules which do not belong to $\mathcal{C}$. We refer to [4, Section 5] for properties of the module E. In particular, we recall that the indecomposable summands of E do not generally form sections (or even left sections) in the Auslander–Reiten components containing them. The following lemma shows however that they form subquivers with similar (though weaker) properties.

Lemma. *Let $E_0, E_1 \in \operatorname{add} E$ be indecomposables. Then*:

(a) *If we have an irreducible morphism $E_0 \longrightarrow X$ with X indecomposable and E_0 non-injective, then $X \in \operatorname{add} E$ or $\tau X \in \operatorname{add} E$.*
(b) *If we have an irreducible morphism $X \longrightarrow E_0$, with X indecomposable and E_0 non-injective, then $X \in \operatorname{add} E$ or $\tau^{-1} X \in \operatorname{add} E$.*
(c) *Let $s, t \geqslant 0$ and $\tau^s E_0 \longrightarrow \tau^{-t} E_1$ be an irreducible morphism. If E_0 and E_1 are non-injective, then $s, t \in \{0, 1\}$ and at least one of them is zero.*
(d) *If we have a path of irreducible morphisms between indecomposables of the form*

$$\tau^{-s} E_1 = X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_t = E_0$$

with $s, t \geqslant 0$ and $X_i \notin \operatorname{add} E$ for all i with $0 < i < t$, then $s = 0$ and moreover, if $t \geqslant 1$, then E_1 is injective.

Proof. (a) By hypothesis, $\tau^{-1} E_0 \notin \mathcal{C}$. Assume $X \in \mathcal{C}$. If $X \notin \operatorname{add} E$, then in particular, X is non-injective and $\tau^{-1} X \in \mathcal{C}$, contradicting the fact that there exists an irreducible morphism $\tau^{-1} E_0 \longrightarrow \tau^{-1} X$. Thus $X \in \mathcal{C}$ implies $X \in \operatorname{add} E$. If $X \notin \mathcal{C}$, then $\tau X \longrightarrow E_0$ yields $\tau X \in \mathcal{C}$ hence $\tau^{-1}(\tau X) = X \notin \mathcal{C}$ gives $\tau X \in \operatorname{add} E$.

(b) If X is injective, then $X \in \operatorname{add} E$. Otherwise, apply (a) to the irreducible morphism $E_0 \longrightarrow \tau^{-1} X$.

(c) If $t \geqslant 1$ and $s \geqslant 1$, we have first $\tau^{-1} E_1 \notin \mathcal{C}$ (because it is a successor of $\tau^{-1} E_1$) and also an irreducible morphism $\tau^{-t} E_1 \longrightarrow \tau^{s-1} E_0$. Hence $\tau^{s-1} E_0 \notin \mathcal{C}$ which contradicts the fact that $\tau^{s-1} E_0 \in \mathcal{C}$ because it precedes E_0. Therefore, $t = 0$ or $s = 0$. Suppose $t = 0$, we have an arrow $\tau^s E_0 \longrightarrow E_1$. Applying (b), we have either $\tau^s E_0 \in \operatorname{add} E$ (hence $s = 0$) or $\tau^{s-1} E_0 \in \operatorname{add} E$ (hence $s = 1$). Suppose now $s = 0$, we have an arrow $E_0 \longrightarrow \tau^{-t} E_1$. Applying (a), we have either $\tau^{-t} E_1 \in \operatorname{add} E$ (hence $t = 0$) or $\tau^{-t+1} E_1 \in \operatorname{add} E$ (hence $t = 1$).

(d) Assume $s \geqslant 1$, then E_1 is not injective and $\tau^{-1} E_1 \notin \mathcal{C}$. But then $\tau^{-s} E_1 \notin \mathcal{C}$ so $X_i \notin \mathcal{C}$ for all i and this contradicts $E_0 \in \mathcal{C}$. Therefore $s = 0$. Assume now that $t \geqslant 1$ and that E_1 is non-injective. Applying (a), $X_1 \notin \operatorname{add} E$ implies $\tau X_1 \in \operatorname{add} E$, hence $X_1 = \tau^{-1}(\tau X_1) \notin \mathcal{C}$. Therefore $E_0 \notin \mathcal{C}$, a contradiction. $\square$

2.3. Notice that, by [16, (3.3), (3.7)] (or 1.5 above), $\tau_A^{-1}E \oplus F$ is the direct sum of a complete set of representatives of the isomorphism classes of indecomposable Ext-projectives in $\operatorname{add}\mathcal{C}^c$. The following lemma is simply an adaptation to our situation of Smalø's theorem [20].

Lemma. *Let $\mathcal{C}$ be a full subcategory of $\operatorname{ind} A$ closed under predecessors. The following conditions are equivalent:*

(a) $\operatorname{add}\mathcal{C}$ *is contravariantly finite.*
(b) $\operatorname{add}\mathcal{C} = \operatorname{Cogen} E_0$, *with $E_0 \in \operatorname{add} E$.*
(c) $\operatorname{add}\mathcal{C} = \operatorname{Cogen} E$.
(d) $\operatorname{add}\mathcal{C}^c$ *is covariantly finite.*
(e) $\operatorname{add}\mathcal{C}^c = \operatorname{Gen}(\tau_A^{-1}E_0 \oplus F_0)$, *where $E_0 \in \operatorname{add} E$ and $F_0 \in \operatorname{add} F$.*
(f) $\operatorname{add}\mathcal{C}^c = \operatorname{Gen}(\tau_A^{-1}E \oplus F)$.

Proof. The equivalence of (a), (b) and (d) follows directly from [20]. Also, (c) implies (b) trivially. Assume (b). Since $E_0 \in \operatorname{add} E$, then $\operatorname{add}\mathcal{C} = \operatorname{Cogen} E_0 \subset \operatorname{Cogen} E$. On the other hand, $E \in \operatorname{add}\mathcal{C}$, and $\mathcal{C}$ is closed under predecessors. Hence, $\operatorname{Cogen} E \subset \operatorname{add}\mathcal{C}$. This shows (c). The equivalence with the remaining conditions follows by duality. $\square$

2.4. We recall from [13, Section 5] that an Ext-injective E_0 in a full additive subcategory $\mathcal{X}$ of $\operatorname{mod} A$ is a *strong Ext-injective* provided $\operatorname{Ext}_A^i(-, E_0)|_{\mathcal{X}} = 0$ for all $i > 0$ (or, equivalently, if $\mathcal{X} \subset {}^{\perp}E_0$).

Lemma. *Let $\mathcal{C}$ be a full subcategory of $\operatorname{ind} A$ closed under predecessors and E_0 be an indecomposable Ext-injective in $\operatorname{add}\mathcal{C}$. Then E_0 is a strong Ext-injective.*

Proof. We prove by induction on i that $\operatorname{Ext}_A^i(X, E_0) = 0$ for all $i > 0$ and all $X \in \mathcal{C}$. If $i = 1$, there is nothing to prove. Assume the result for $i - 1$ and let $X \in \mathcal{C}$. Consider the short exact sequence

$$0 \longrightarrow K \longrightarrow P \xrightarrow{p} X \longrightarrow 0$$

where p is a projective cover. Since $\mathcal{C}$ is closed under predecessors, then $K \in \operatorname{add}\mathcal{C}$. The induction hypothesis then implies that $\operatorname{Ext}_A^i(X, E_0) \cong \operatorname{Ext}_A^{i-1}(K, E_0) = 0$. $\square$

2.5.

Lemma. *Let $\mathcal{C}$ be a full subcategory of $\operatorname{ind} A$ closed under predecessors. Then:*

(a) $\operatorname{add}(\mathcal{C} \cap \mathcal{C}^{\perp}) = \operatorname{add} E$.
(b) $\operatorname{add}(\mathcal{C}^c \cup \mathcal{E}) = \mathcal{C}^{\perp}$.

Proof. (a) Let $X \in \mathcal{C} \cap \mathcal{C}^{\perp}$. If $X \notin \operatorname{add} E$, then $\tau_A^{-1}X \in \mathcal{C}$. Since $\operatorname{Ext}_A^1(\tau_A^{-1}X, X) \neq 0$, then $X \notin \mathcal{C}^{\perp}$, a contradiction. Therefore, $\operatorname{add}(\mathcal{C} \cap \mathcal{C}^{\perp}) \subset \operatorname{add} E$. On the other hand, $E \in \operatorname{add}\mathcal{C}$ implies $\operatorname{add} E \subset \operatorname{add}\mathcal{C}$. Also, because of 2.4, $E \in \mathcal{C}^{\perp}$ and therefore $\operatorname{add} E \subset \operatorname{add}(\mathcal{C} \cap \mathcal{C}^{\perp})$.

(b) Let $X \in \mathcal{C}^c$. If $X \notin \mathcal{C}^{\perp}$, there exist $i > 0$ and $M \in \mathcal{C}$ such that $\operatorname{Ext}_A^i(M, X) \neq 0$. By 2.1, there exists a path $X \rightsquigarrow M$. Since $\mathcal{C}$ is closed under predecessors, we infer that $X \in \mathcal{C}$, a contradiction. This shows that $\mathcal{C}^c \subset \mathcal{C}^{\perp}$. Since $E \in \mathcal{C}^{\perp}$, by 2.4, we deduce that $\operatorname{add}(\mathcal{C}^c \cup \mathcal{E}) \subset \mathcal{C}^{\perp}$. Applying (a), we get $\operatorname{add}(\mathcal{C}^c \cup \mathcal{E}) = \operatorname{add}(\mathcal{C}^c \cup (\mathcal{C} \cap \mathcal{C}^{\perp})) = \operatorname{add}(\mathcal{C}^c \cup \mathcal{C}^{\perp}) = \mathcal{C}^{\perp}$. $\square$

2.6. While $\mathcal{C}$ closed under predecessors implies that $\mathcal{C}^c$ is closed under successors, the right orthogonal $\mathcal{C}^{\perp}$ is usually not closed under successors. Indeed, we show that this is the case if and only if $\operatorname{pgd}\mathcal{C} \leqslant 1$ (that is, $\mathcal{C} \subset \mathcal{L}_A$).

Corollary. *Assume that $\mathcal{C}$ is a full subcategory of* $\operatorname{ind} A$ *closed under predecessors. Then $\mathcal{C}^\perp$ is closed under successors if and only if* $\operatorname{pgd}\mathcal{C} \leqslant 1$.

Proof. Assume first that $\operatorname{pgd}\mathcal{C} \leqslant 1$. Let $M \in \mathcal{C}^\perp$ be indecomposable and assume that we have a path $M \rightsquigarrow N$ in $\operatorname{ind} A$. By 2.5, either $M \in \mathcal{C}^c$, and then $N \in \mathcal{C}^c$, or else $M \in \mathcal{E}$ and then, applying [4, (6.3)], we get either $N \in \mathcal{E}$ or else $N \in \mathcal{C}^c$.

Conversely, assume that $\operatorname{pgd}\mathcal{C} > 1$. Then there exists $X \in \mathcal{C}$ such that $\operatorname{pd} X > 1$. In particular, X is non-projective, and there exist an injective module I and a non-zero morphism $I \longrightarrow \tau_A X$. Clearly, $\tau_A X \in \mathcal{C}$ because $X \in \mathcal{C}$. Also, $\tau_A X \notin \mathcal{E}$ because $\tau_A^{-1}(\tau_A X) = X \in \mathcal{C}$. Therefore, $\tau_A X \notin \mathcal{C}^\perp$. But $I \in \mathcal{E} \subset \mathcal{C}^\perp$ shows that $\mathcal{C}^\perp$ is not closed under successors. □

2.7.

Corollary. *Assume that $\mathcal{C}$ is a full subcategory of* $\operatorname{ind} A$ *closed under predecessors. Then* $\operatorname{add}\mathcal{C}$ *is contravariantly finite if and only if $\mathcal{C}^\perp$ is covariantly finite.*

Proof. By 2.3, $\operatorname{add}\mathcal{C}$ is contravariantly finite if and only if $\operatorname{add}\mathcal{C}^c$ is covariantly finite. In view of 2.5 and [6, (5.7)], this is the case if and only if $\mathcal{C}^\perp$ is covariantly finite. □

2.8.

Lemma. *Assume that $\mathcal{C}$ is a full subcategory of* $\operatorname{ind} A$ *closed under predecessors. Then* $\operatorname{add}\mathcal{C}$ *is contravariantly finite if and only if* $\operatorname{add}\mathcal{C} = \operatorname{Supp}(-, E)$.

Proof. *Necessity.* Since $\operatorname{add}\mathcal{C}$ is contravariantly finite, we have $\operatorname{add}\mathcal{C} = \operatorname{Cogen} E$ (by 2.3). The statement follows from the inclusions

$$\operatorname{Cogen} E \subset \operatorname{Supp}(-, E) \subset \operatorname{add}\mathcal{C}.$$

Sufficiency. The hypothesis says that $\operatorname{Supp}(-, E)$ is closed under predecessors. Therefore, by [6, (2.1)], $\operatorname{Supp}(-, E) = \operatorname{Cogen} E$. The result follows from $\operatorname{Supp}(-, E) = \operatorname{Cogen} E \subset \operatorname{add}\mathcal{C} = \operatorname{Supp}(-, E)$. □

Remark. The previous lemma could be formulated otherwise. By [6, (2.2)], if $\mathcal{C}$ is closed under predecessors, then the above lemma says that the following conditions are equivalent:

(a) $\operatorname{add}\mathcal{C}$ is contravariantly finite.
(b) $\operatorname{add}\mathcal{C} = \operatorname{Supp}(-, E)$.
(c) There exists an A-module L such that $\operatorname{add}\mathcal{C} = \operatorname{Supp}(-, L)$ and $\operatorname{Hom}_A(\tau_A^{-1} L, L) = 0$.
(d) There exists an A-module L such that $\operatorname{add}\mathcal{C} = \operatorname{Supp}(-, L)$ and there is no path of the form $\tau_A^{-1} L_i \rightsquigarrow L_j$ with L_i and L_j indecomposable summands of L.

2.9. We are now able to prove our first theorem, which characterises the case where our subcategory $\mathcal{C}$ is resolving. Here it is important to note that, since $\mathcal{C}$ is closed under predecessors, then it is resolving if and only if it contains all the projectives in $\operatorname{mod} A$.

Theorem. *Assume that $\mathcal{C}$ is a full subcategory of* $\operatorname{ind} A$ *which is closed under predecessors. The following conditions are equivalent*:

(a) $\operatorname{add}\mathcal{C}$ *is contravariantly finite and resolving.*
(b) $\mathcal{C}^\perp$ *is covariantly finite and* $\operatorname{add}\mathcal{C} = {}^\perp(\mathcal{C}^\perp)$.
(c) *E is a cotilting module.*
(d) $\operatorname{add}\mathcal{C} = \operatorname{Supp}(-, E)$ *and E is sincere.*

Moreover, if this is the case, then $\operatorname{add}\mathcal{C} = {}^\perp E$.

Proof. (a) *implies* (b). This follows from 2.7 and [13, (3.3)].

(b) *implies* (a). This follows again from 2.7 and the obvious observation that ${}^{\perp}(\mathcal{C}^{\perp})$ is resolving.

(a) *implies* (c). We claim that, for every A-module M, there exists an exact sequence

$$0 \longrightarrow X_1 \longrightarrow X_0 \longrightarrow M \longrightarrow 0$$

with $X_0, X_1 \in \operatorname{add}\mathcal{C}$. We may, of course, suppose that M is indecomposable and not in $\mathcal{C}$. Let $p : P \longrightarrow M$ be a projective cover. Since $\operatorname{add}\mathcal{C}$ is resolving, then $P \in \operatorname{add}\mathcal{C}$. Since $\mathcal{C}$ is closed under predecessors, then $\operatorname{Ker} p \in \operatorname{add}\mathcal{C}$. The sequence

$$0 \longrightarrow \operatorname{Ker} p \longrightarrow P \xrightarrow{p} M \longrightarrow 0$$

is the required one.

Applying Auslander–Reiten's theorem 1.3, there exists a cotilting module T such that $\operatorname{add}\mathcal{C} = {}^{\perp}T$. By the dual of 1.5, $\operatorname{add} T$ is the subcategory of Ext-injectives in $\operatorname{add}\mathcal{C}$, that is, $\operatorname{add} T = \operatorname{add} E$. Thus, E is a cotilting module. Also, $\operatorname{add}\mathcal{C} = {}^{\perp}E$.

(c) *implies* (a). By the dual of Happel's theorem 1.2, ${}^{\perp}E \subset \operatorname{Cogen} E$. Clearly, $\operatorname{Cogen} E \subset \operatorname{add}\mathcal{C}$. By 2.4, we also have $\operatorname{add}\mathcal{C} \subset {}^{\perp}E$. Therefore, $\operatorname{add}\mathcal{C} = {}^{\perp}E$. In particular, $\operatorname{add}\mathcal{C}$ is contravariantly finite.

(a) *is equivalent to* (d). This follows from 2.8, using that E is sincere if and only if every indecomposable projective lies in $\operatorname{Supp}(-, E)$, which is the case if and only if $\operatorname{add}\mathcal{C}$ is resolving. □

Remark. It is useful to observe that, if $\operatorname{add}\mathcal{C}$ is contravariantly finite and resolving, then E is a cotilting module of injective dimension at most one. This indeed follows from the fact that $\tau^{-1}E \in \operatorname{add}\mathcal{C}^c$ and $A_A \in \operatorname{add}\mathcal{C}$ yield $\operatorname{Hom}_A(\tau_A^{-1}E, A) = 0$.

2.10. We end this section with the following example which originates from the theory of m-clusters (see [5]).

Example. Let A be any Artin algebra. For any $m > 0$, we define $\mathcal{L}_A^{(m)}$ to be the full subcategory of $\operatorname{ind} A$ consisting of all indecomposable A-modules M such that $L \rightsquigarrow M$, then $\operatorname{pd} L \leqslant m$ (thus $\mathcal{L}_A^{(1)} = \mathcal{L}_A$). Clearly, $\mathcal{L}_A^{(m)}$ is closed under predecessors.

Let now H be a hereditary algebra over an algebraically closed field, and A be the m-replicated algebra of H, that is,

$$A = \begin{bmatrix} H_0 & 0 & 0 & \cdots & 0 & 0 \\ Q_1 & H_1 & 0 & \cdots & 0 & 0 \\ 0 & Q_2 & H_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & Q_m & H_m \end{bmatrix}$$

where $H_i = H$ and $Q_i = \mathrm{D}H$ for all i, and all the remaining coefficients are zero. The addition is the usual addition of matrices and the multiplication is induced from the canonical isomorphisms

$$H \otimes_H \mathrm{D}H \cong_H \mathrm{D}H_H \cong \mathrm{D}H \otimes_H H$$

and the zero morphism $\mathrm{D}H \otimes_H \mathrm{D}H \longrightarrow 0$.

We claim that $\operatorname{add}\mathcal{L}_A^{(m)}$ is contravariantly finite and resolving. Indeed, let, for any $k \geqslant 0$, Σ_k consist of the indecomposable summands of the kth-cosyzygy $\Omega_A^{-k}H$ of the indecomposable projective A-modules corresponding to the idempotents of H_0, then it is shown in [5, Corollary 18], that, if

$M \in \operatorname{ind} A$ is not projective–injective, then $M \in \mathcal{L}_A^{(k)}$ if and only if M precedes a module in Σ_k. Now, using the description of $\operatorname{mod} A$ in [5, (3.1)], it follows that $\mathcal{L}_A^{(m)}$ consists of the predecessors of Σ_m together with all the projective–injectives A-modules. Therefore, $\mathcal{L}_A^{(m)}$ is resolving. Let now $M \in \mathcal{L}_A^{(m)}$ then it follows from the same description that an injective envelope $M \hookrightarrow I$ factors through Σ_m. Therefore, $\mathcal{L}_A^{(m)}$ is also contravariantly finite.

This implies that E (which, in this case, is the direct sum of the modules on Σ_m with all the projective–injective A-modules) is a cotilting module. Since $\operatorname{gl.dim} A < \infty$, it is also tilting (by 1.7). Hence, by the main result of [5], it corresponds to an m-cluster. Notice that, if $m = 1$, an m-cluster is simply a cluster.

3. Tilting modules

3.1.

Lemma. *Assume that $\mathcal{C}$ is a full subcategory of* $\operatorname{ind} A$*, closed under predecessors. Then the Ext-projectives of $\mathcal{C}^\perp$ are the objects of* $\operatorname{add}(E \oplus F)$.

Proof. We claim that, if X is an indecomposable Ext-projective in $\mathcal{C}^\perp$, then $X \in \operatorname{add} E$ or $X \in \operatorname{add} F$. Suppose $X \notin \operatorname{add} E$. By 2.5, $X \notin \mathcal{C}$. Suppose X is not projective. Since $X \in \mathcal{C}^c$ and is Ext-projective in $\mathcal{C}^\perp$, then it is also Ext-projective in $\mathcal{C}^c$. Hence $\tau_A X \in \mathcal{C}$. But then $\tau_A^{-1}(\tau_A X) = X \in \mathcal{C}^c$ gives $\tau_A X \in \operatorname{add} E$. Now, $\operatorname{Ext}_A^1(X, \tau_A X) \neq 0$ gives a contradiction to the Ext-projectivity of X in $\mathcal{C}^\perp$. This shows that X is projective. Since $X \notin \mathcal{C}$, we have $X \in \operatorname{add} F$. This establishes our claim.

On the other hand, $E \in \operatorname{add} \mathcal{C}$ implies that, for any $i > 0$ and every $Y \in \mathcal{C}^\perp$, we have $\operatorname{Ext}_A^i(E, Y) = 0$. In particular, E is Ext-projective in $\mathcal{C}^\perp$. This completes the proof. $\square$

3.2. Following [6], we denote by $\operatorname{Pred} E$ the full subcategory of $\operatorname{ind} A$ consisting of all the predecessors of the indecomposable summands of E (that is, of the objects in $\mathcal{E}$).

Lemma. *Assume that $\mathcal{C}$ is a full subcategory of* $\operatorname{ind} A$*, closed under predecessors. Then $\mathcal{C}^\perp = E^\perp$ implies $\mathcal{C} =$* $\operatorname{Pred} E$.

Proof. Since $E \in \operatorname{add} \mathcal{C}$, then $\operatorname{Pred} E \subset \operatorname{add} \mathcal{C}$. We claim that, if $X \in \mathcal{C}$, then $X \in \operatorname{Pred} E$. We may assume that $X \notin \operatorname{add} E$. Since, by our hypothesis, $\mathcal{C}^\perp = E^\perp$, then it follows from 2.5 that $X \notin E^\perp$. Therefore there exists an $i > 0$ such that $\operatorname{Ext}_A^i(E, X) \neq 0$, thus, by 2.1, there exist an indecomposable $E_0 \in \operatorname{add} E$ and a path $X \rightsquigarrow E_0$. In particular, $X \in \operatorname{Pred} E$. $\square$

Remark. It is shown in [6] and [4], respectively, that, if $\mathcal{C} = \mathcal{L}_A$ or, more generally, if $\operatorname{pgd} \mathcal{C} \leqslant 1$, then the condition $\mathcal{C} = \operatorname{Pred} E$ is equivalent to having $\operatorname{add} \mathcal{C}$ contravariantly finite. While we show in 3.5 below that $\operatorname{add} \mathcal{C}$ contravariantly finite implies that $\mathcal{C} = \operatorname{Pred} E$, the following example shows that the converse is not true in general.

Notation. Here, and in the sequel, when dealing with a bound quiver algebra, we denote by P_x, I_x and S_x respectively, the indecomposable projective, the indecomposable injective and the simple modules corresponding to the point x of the quiver.

Example. Let A be given by the quiver

$$\begin{tikzcd} \underset{1}{\bullet} & \underset{2}{\bullet} \arrow[l, "\delta"'] & \underset{3}{\bullet} \arrow[l, shift right, "\beta"'] \arrow[l, shift left, "\gamma"] & \underset{4}{\bullet} \arrow[l, "\alpha"'] \end{tikzcd}$$

bound by $\alpha\beta = 0$, $\beta\delta = 0$. Then $\Gamma(\operatorname{mod} A)$ contains a tube in which lies the unique projective–injective indecomposable $P_4 = I_1$. Let $\mathcal{C}$ consist of the indecomposables lying in this tube or in the postprojective component. Clearly, $\mathcal{C}$ is closed under predecessors and $E = P_4$. Also, $\operatorname{add}\mathcal{C} = \operatorname{Pred} P_4$ but it is not contravariantly finite.

3.3.

Lemma. *Assume that $\mathcal{C}$ is a full subcategory of* $\operatorname{ind} A$*, closed under predecessors and let $T = E \oplus F$. Then $\operatorname{add}\mathcal{C} \subset {}^{\perp}T$. In particular, $T = E \oplus F$ is an auto-orthogonal module.*

Proof. Let $M \in \mathcal{C}$. By 2.4, we have $M \in {}^{\perp}E$. Hence, if $M \notin {}^{\perp}T$, then there exist $i > 0$ and an indecomposable summand F_0 of F such that $\operatorname{Ext}^i_A(M, F_0) \neq 0$. But then, by 2.1, there exists a path $F_0 \rightsquigarrow M$. Since $M \in \mathcal{C}$, this gives $F_0 \in \mathcal{C}$, a contradiction. This shows that $M \in {}^{\perp}T$, and thus $\operatorname{add}\mathcal{C} \subset {}^{\perp}T$. In particular, $E \in \operatorname{add}\mathcal{C}$ yields $E \in {}^{\perp}T$. Hence T is auto-orthogonal. $\square$

3.4.

Lemma. *Assume that $\mathcal{C}$ is a full subcategory of* $\operatorname{ind} A$ *closed under predecessors and such that* $\operatorname{pgd}\mathcal{C} < \infty$*. If $T = E \oplus F$ is a tilting A-module, then*

(a) $\mathcal{C}^{\perp} = T^{\perp}$; *and*
(b) $\operatorname{add}\mathcal{C}$ *is contravariantly finite.*

Proof. (a) Since $E \in \operatorname{add}\mathcal{C}$, then $\mathcal{C}^{\perp} \subset E^{\perp}$. Since F is projective, we have $E^{\perp} = T^{\perp}$, so that $\mathcal{C}^{\perp} = T^{\perp}$. Conversely, assume that $X \in T^{\perp}$ is indecomposable. By Happel's theorem 1.2, $T^{\perp} \subset \operatorname{Gen} T$, so $X \in \operatorname{Gen} T$. By 1.4, there exists an exact sequence

$$0 \longrightarrow K_0 \longrightarrow T_0 \xrightarrow{f_0} X \longrightarrow 0$$

with $K_0 \in T^{\perp} \subset \operatorname{Gen} T$ and $T_0 \in \operatorname{add} T$. Inductively, we get an exact sequence

$$0 \longrightarrow K_{d-1} \longrightarrow T_{d-1} \xrightarrow{f_{d-1}} \cdots \longrightarrow T_1 \xrightarrow{f_1} T_0 \xrightarrow{f_0} X \longrightarrow 0$$

where $d = \operatorname{pgd}\mathcal{C} < \infty$, and such that $T_i \in \operatorname{add} T$, for all i, and $K_i = \operatorname{Ker} f_i$ lies in $T^{\perp} \subset \operatorname{Gen} T$ for all i.

Let now $M \in \mathcal{C}$. By 3.3, we have $M \in {}^{\perp}T$. Therefore, applying the functor $\operatorname{Hom}_A(M, -)$ to each of the sequences

$$0 \longrightarrow K_i \longrightarrow T_i \longrightarrow K_{i-1} \longrightarrow 0$$

where $0 \leqslant i < d$, and $K_{-1} = X$ yields, for each $j > 0$, an isomorphism

$$\operatorname{Ext}^j_A(M, K_{i-1}) \cong \operatorname{Ext}^{j+1}_A(M, K_i).$$

Therefore

$$\operatorname{Ext}^j_A(M, X) \cong \operatorname{Ext}^{j+1}_A(M, K_0) \cong \cdots \cong \operatorname{Ext}^{j+d}_A(M, K_{d-1}) = 0$$

for all $j > 0$, because $\operatorname{pd} M \leqslant d$. Therefore, $X \in \mathcal{C}^{\perp}$.

(b) By Auslander–Reiten's theorem 1.3, T tilting implies $T^{\perp}$ covariantly finite. By (a), $\mathcal{C}^{\perp}$ is covariantly finite. By 2.7, $\operatorname{add}\mathcal{C}$ is contravariantly finite. $\square$

3.5. We are now able to prove the main result of this section.

Theorem. *Assume that $\mathcal{C}$ is a full subcategory of* $\operatorname{ind} A$ *closed under predecessors and such that* $\operatorname{pgd}\mathcal{C} < \infty$. *Then* $\operatorname{add}\mathcal{C}$ *is contravariantly finite and resolving if and only if E is a tilting module. Moreover, if this is the case, then $\mathcal{C}^{\perp} = E^{\perp}$, $\mathcal{C} = \operatorname{Pred} E$ and $\operatorname{add}\mathcal{C} = {}^{\perp}(E^{\perp})$.*

Proof. Assume first that $\operatorname{add}\mathcal{C}$ is contravariantly finite and resolving and let $d = \operatorname{pgd}\mathcal{C} < \infty$. By 2.9, E is a cotilting module. We claim that it is tilting. Observe first that E is an auto-orthogonal module, and $\operatorname{pd} E \leqslant d$. Since $\operatorname{add}\mathcal{C}$ is resolving, we have $A_A \in \operatorname{add}\mathcal{C} = \operatorname{Cogen} E$. By 1.4, we have a short exact sequence

$$0 \longrightarrow A_A \xrightarrow{f^0} E^0 \longrightarrow K^0 \longrightarrow 0$$

with $K^0 \in {}^{\perp}E \subset \operatorname{Cogen} E$, and $E^0 \in \operatorname{add} E$. Inductively, we get an exact sequence

$$0 \longrightarrow A_A \xrightarrow{f^0} E^0 \xrightarrow{f^1} E^1 \longrightarrow \cdots \longrightarrow E^{d-2} \xrightarrow{f^{d-1}} E^{d-1} \longrightarrow K^{d-1} \longrightarrow 0$$

such that $E^i \in \operatorname{add} E$ for all i, and $K^i = \operatorname{Coker} f^i$ lies in ${}^{\perp}E \subset \operatorname{Cogen} E$ for all i.

Applying the functor $\operatorname{Hom}_A(M, -)$ with $M \in {}^{\perp}E$ to each of the sequences

$$0 \longrightarrow K^{i-1} \longrightarrow E^i \longrightarrow K^i \longrightarrow 0$$

where $0 \leqslant i < d$, and $K^{-1} = A_A$ yields, for each $j > 0$, an isomorphism

$$\operatorname{Ext}_A^j\big(M, K^i\big) \cong \operatorname{Ext}_A^{j+1}\big(M, K^{i-1}\big).$$

Hence

$$\operatorname{Ext}_A^j\big(E, K^{d-1}\big) \cong \operatorname{Ext}_A^{j+d}(E, A_A) = 0$$

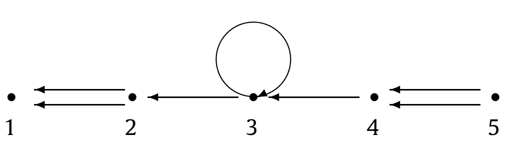
because $\operatorname{pd} E \leqslant d$. Similarly, $\operatorname{Ext}_A^j(K^{d-1}, K^{d-1}) = 0$, for all $j > 0$. This shows that $E \oplus K^{d-1}$ is auto-orthogonal. Since $K^{d-1} \in {}^{\perp}E$ and we have ${}^{\perp}E \subset \operatorname{Cogen} E \subset \operatorname{add}\mathcal{C}$, then $\operatorname{pd} K^{d-1} < \infty$. This proves that $E \oplus K^{d-1}$ is a tilting A-module. Looking at the number of isomorphism classes of indecomposable summands, we deduce that $K^{d-1} \in \operatorname{add} E$. Consequently, E is a tilting A-module.

Assume conversely that E is a tilting A-module. In particular, for every indecomposable projective A-module P, there exists a monomorphism $P \hookrightarrow E^0$, with $E^0 \in \operatorname{add} E$. Therefore, $P \in \operatorname{add}\mathcal{C}$. This shows that $\operatorname{add}\mathcal{C}$ is resolving. Thus, we have $F = 0$. By 3.4 above, we get that $\operatorname{add}\mathcal{C}$ is contravariantly finite and that $\mathcal{C}^{\perp} = T^{\perp} = E^{\perp}$.

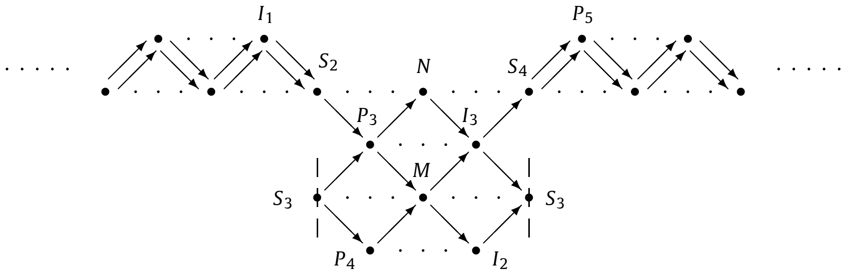
Finally, by 3.2, we have $\mathcal{C} = \operatorname{Pred} E$ and, by 2.9, we have $\operatorname{add}\mathcal{C} = {}^{\perp}(\mathcal{C}^{\perp}) = {}^{\perp}(E^{\perp})$. $\square$

3.6. The statement of the theorem is not true if we drop the condition that $\operatorname{pdg}\mathcal{C} < \infty$. We give an example of a contravariantly finite and resolving subcategory $\mathcal{C}$, where the Ext-injective module E has $\operatorname{pd} E = \infty$ and hence is not a tilting module (but it is, of course, cotilting).

Example. Let A be given by the quiver



bound by $\operatorname{rad}^2 A = 0$. The Auslander–Reiten quiver $\Gamma(\operatorname{mod} A)$ of A has a component Γ of the following shape



where we identify the two copies of S_3, along the vertical dotted lines (note that A is a laura algebra, having Γ as its unique faithful quasi-directed component, see [7]). Let $\mathcal{C} = \operatorname{Pred} P_5$ consist of all the predecessors of the projective indecomposable P_5. Observe that $\mathcal{C}$ contains the components of $\Gamma(\operatorname{mod} A)$ which are identified with the components of the Kronecker algebra given by the points 1 and 2. By definition, $\mathcal{C}$ is closed under predecessors. Moreover, $\operatorname{pgd} \mathcal{C} = \infty$ because $S_3 \in \mathcal{C}$ and $\operatorname{pd} S_3 = \infty$. Here, $E = I_1 \oplus I_2 \oplus I_3 \oplus S_4 \oplus P_5$. By 2.7, E is a cotilting module. However, it is not a tilting module, because $\operatorname{pd} S_4 = \infty$.

4. The general case

4.1. Let $\mathcal{C}$ be a full subcategory of $\operatorname{ind} A$, closed under predecessors. Following [4], we define its *support algebra* ${}_{\mathcal{C}}A$ to be the endomorphism algebra of the direct sum of a full set of representatives of the isomorphism classes of the indecomposable projectives lying in $\mathcal{C}$.

We need some notations. We sometimes consider an algebra A as a category in which the class of objects is a complete set $\{e_1, \dots, e_n\}$ of primitive orthogonal idempotents and the set of morphisms from e_i to e_j is $e_i A e_j$. An algebra B is a *full subcategory* of A if there is an idempotent $e \in A$ which is the sum of some of the distinguished idempotents e_i, such that $B = eAe$. It is *convex* in A if, for any sequence $e_i = e_{i_0}, e_{i_1}, \dots, e_{i_t} = e_j$ of objects of A such that $e_{i_l} A e_{i_{l+1}} \neq 0$ (with $0 \leqslant l < t$) and e_i, e_j objects in B, then all e_{i_l} lie in B. We now collect some properties of the support algebra.

Lemma. *Let $\mathcal{C}$ be a full subcategory of* $\operatorname{ind} A$, *closed under predecessors.*

(a) *${}_{\mathcal{C}}A$ is a full convex subcategory of A, closed under successors.*
(b) *Any indecomposable A-module lying in $\mathcal{C}$ has a canonical structure of indecomposable ${}_{\mathcal{C}}A$-module.*
(c) $\operatorname{add} \mathcal{C}$ *is resolving in* $\operatorname{mod}_{\mathcal{C}} A$.
(d) *For any indecomposable ${}_{\mathcal{C}}A$-module X, we have* $\operatorname{pd}_{{}_{\mathcal{C}}A} X = \operatorname{pd}_A X$ *and* $\operatorname{id}_{{}_{\mathcal{C}}A} X \leqslant \operatorname{id}_A X$. *In particular,* $\operatorname{gl.dim}_{\mathcal{C}} A \leqslant \operatorname{gl.dim} A$.

Proof. (a) and (b) are straightforward, (c) follows from the facts that $\mathcal{C}$ is closed under predecessors and that any indecomposable projective ${}_{\mathcal{C}}A$-module lies in $\mathcal{C}$. Finally, (d) follows from the facts that, because ${}_{\mathcal{C}}A$ is convex in A then, for any two ${}_{\mathcal{C}}A$-modules L and M, we have $\operatorname{Ext}^i_{{}_{\mathcal{C}}A}(L, M) \cong \operatorname{Ext}^i_A(L, M)$ for all $i \geqslant 0$ and the observation that any module in a projective resolution of L in $\operatorname{mod} A$ is also a projective ${}_{\mathcal{C}}A$-module. □

4.2. We are now able to state, and to prove, the first main result of this paper.

Theorem. *Let $\mathcal{C}$ be a full subcategory of* $\operatorname{ind} A$, *closed under predecessors. The following conditions are equivalent*:

(a) $\operatorname{add}\mathcal{C}$ *is contravariantly finite.*
(b) $\mathcal{C}^{\perp}$ *is covariantly finite.*
(c) E *is a cotilting* ${}_{\mathcal{C}}A$*-module.*
(d) $\operatorname{add}\mathcal{C} = \operatorname{Supp}(-, E)$.
(e) *Any morphism* $f : L \longrightarrow M$ *with* $L \in \mathcal{C}$ *and* M *indecomposable not in* $\mathcal{C}$ *factors through* $\operatorname{add} E$.

Proof. By 2.7, (a) is equivalent to (b) and, by 2.8, (a) is equivalent to (d). We now show the equivalence of (a) and (c). Observe that $\operatorname{add}\mathcal{C}$ is contravariantly finite in $\operatorname{mod} A$ if and only if $\operatorname{add}\mathcal{C} = \operatorname{Cogen} E$ by 2.3, and this is the case if and only if $\operatorname{add}\mathcal{C}$ is contravariantly finite in $\operatorname{mod}_{\mathcal{C}} A$, which, because of 4.1(c) and 2.9, happens if and only if E is a cotilting ${}_{\mathcal{C}}A$-module.

We now prove that (a) implies (e). By the dual of 1.4, there exists a short exact sequence

$$0 \longrightarrow L \xrightarrow{f_0} E_0 \longrightarrow K \longrightarrow 0$$

in $\operatorname{mod}_{\mathcal{C}} A$, where f_0 is a left minimal $\operatorname{add} E$-approximation of L and K belongs to the left orthogonal ${}^{\perp}_{\mathcal{C}}E$ of E in $\operatorname{mod}_{\mathcal{C}} A$. Since E is a cotilting ${}_{\mathcal{C}}A$-module, then, by Happel's theorem 1.2, we have

$${}^{\perp}_{\mathcal{C}}E \subset \operatorname{Cogen} E = \operatorname{add}\mathcal{C}.$$

In particular, $K \in \operatorname{add}\mathcal{C}$. Applying now $\operatorname{Hom}_A(-, M)$ to the above sequence (considered as an exact sequence in $\operatorname{mod} A$), we get an exact sequence

$$0 \longrightarrow \operatorname{Hom}_A(K, M) \longrightarrow \operatorname{Hom}_A(E_0, M) \longrightarrow \operatorname{Hom}_A(L, M) \longrightarrow \operatorname{Ext}^1_A(K, M) \longrightarrow \cdots.$$

We claim that $\operatorname{Ext}^1_A(K, M) = 0$. Indeed, if this is not the case, then there exists an indecomposable summand K' of K such that $\operatorname{Ext}^1_A(K', M) \neq 0$, and this implies the existence of a path $M \rightsquigarrow K'$. Since $K' \in \mathcal{C}$, we infer that $M \in \mathcal{C}$, a contradiction which establishes our claim. This implies that

$$\operatorname{Hom}_A(f_0, M) : \operatorname{Hom}_A(E_0, M) \longrightarrow \operatorname{Hom}_A(L, M)$$

is surjective. Hence there exists $g : E_0 \longrightarrow M$ such that $f = g \circ f_0$. This completes the proof of (e).

Conversely, assume that (e) holds. In order to prove that $\operatorname{add}\mathcal{C}$ is contravariantly finite, it suffices to show that $\operatorname{add}\mathcal{C} = \operatorname{Cogen} E$, and, for this, we just have to prove that any $L \in \mathcal{C}$ is cogenerated by E. Let $j : L \hookrightarrow I$ be an injective envelope. We can decompose I in the form $I = I_1 \oplus I_2$, where $I_1 \in \operatorname{add}\mathcal{C}$ while I_2 collects those indecomposable summands of I which do not belong to $\mathcal{C}$. We may then write j as

$$j = \begin{bmatrix} j_1 \\ j_2 \end{bmatrix} : L \longrightarrow I_1 \oplus I_2.$$

By hypothesis, $j_2 : L \longrightarrow I_2$ factors through $\operatorname{add} E$, therefore there exist $f_2 : L \longrightarrow E_2$, $g_2 : E_2 \longrightarrow I_2$, with $E_2 \in \operatorname{add} E$ such that $j_2 = g_2 f_2$. This shows that j factors through $I_1 \oplus E_2$ which belongs to $\operatorname{add} E$ (because any injective in $\mathcal{C}$ lies in $\operatorname{add} E$). Furthermore, the morphism $\left[\begin{smallmatrix} j_1 \\ f_2 \end{smallmatrix}\right]$ from L to $I_1 \oplus E_2$ is a monomorphism, because so is j. The proof is now finished. $\square$

4.3. As a consequence of this theorem (and [11]), we get a family of examples of contravariantly finite subcategories: if $\mathcal{C}$ is closed under predecessors and $\operatorname{add}\mathcal{C}$ is an abelian exact subcategory, then $\operatorname{add}\mathcal{C}$ is contravariantly finite.

Example. We recall from [11] that the additive subcategory $\operatorname{add}\mathcal{C}$ is called *abelian exact* if it is abelian and the inclusion functor $\operatorname{add}\mathcal{C} \hookrightarrow \operatorname{mod} A$ is exact. If $\mathcal{C}$ is closed under predecessors and $\operatorname{add}\mathcal{C}$ is abelian exact then, by the main result of [11],

$$A \cong \begin{bmatrix} {}_{\mathcal{C}}A & 0 \\ M & B \end{bmatrix}$$

where M is a hereditary injective ${}_{\mathcal{C}}A$-module and $\operatorname{add}\mathcal{C} \cong \operatorname{mod}_{\mathcal{C}} A$.

As a direct consequence, $\operatorname{add}\mathcal{C}$ is contravariantly finite: indeed, $\operatorname{add}\mathcal{C} \cong \operatorname{mod}_{\mathcal{C}} A$ is cogenerated by the minimal injective cogenerator of $\operatorname{mod}_{\mathcal{C}} A$ (which, when considered as an A-module, is equal to E).

4.4. In the case where the projective global dimension of $\mathcal{C}$ is finite, we obtain our second main theorem.

Theorem. *Let $\mathcal{C}$ be a full subcategory of* $\operatorname{ind} A$*, closed under predecessors and such that* $\operatorname{pgd}(\mathcal{C}) < \infty$*. The following conditions are equivalent*:

(a) $\operatorname{add}\mathcal{C}$ *is contravariantly finite.*
(b) E *is a tilting* ${}_{\mathcal{C}}A$*-module.*
(c) $T = E \oplus F$ *is a tilting A-module.*

Moreover, in this case, $\mathcal{C}^{\perp} = T^{\perp} = E^{\perp}$ *and* $\mathcal{C} = \operatorname{Pred} E$.

Proof. It follows from 4.1(c) and 3.5 that (a) implies (b). Now assume (b). By 3.3, the module $T = E \oplus F$ is auto-orthogonal in $\operatorname{mod} A$. Also, $\operatorname{pd} T = \operatorname{pd} E < \infty$. If P is an indecomposable projective A-module, and P lies in $\mathcal{C}$, then there is an exact sequence

$$0 \longrightarrow P \longrightarrow E^0 \longrightarrow E^1 \longrightarrow \cdots \longrightarrow E^s \longrightarrow 0$$

with $E^i \in \operatorname{add} E \subset \operatorname{add} T$. If $P \notin \mathcal{C}$, then $P \in \operatorname{add} F \subset \operatorname{add} T$. Thus, T is a tilting module and we have shown (c). Finally, assume (c). Then (a) follows from 3.4 which also gives $\mathcal{C}^{\perp} = T^{\perp} = E^{\perp}$. By 3.2, we deduce that $\mathcal{C} = \operatorname{Pred} E$. $\square$

Remarks. Applying our theorems to the case where $\mathcal{C} = \mathcal{L}_A$, we get Theorem (A) of [9] and parts of Theorem (A) of [6]. Applying them to the case where $\operatorname{pdg}(\mathcal{C}) \leqslant 1$, we get parts of Theorem (8.2) and Corollary (8.4) of [4]. Our theorem may thus be considered as a generalisation of these results.

4.5.

Corollary. *Let $\mathcal{C}$ be a full subcategory of* $\operatorname{ind} A$*, closed under predecessors and assume that* $\operatorname{gl.dim} A < \infty$*. Then* $\operatorname{add}\mathcal{C}$ *is contravariantly finite if and only if $T = E \oplus F$ is a cotilting A-module.*

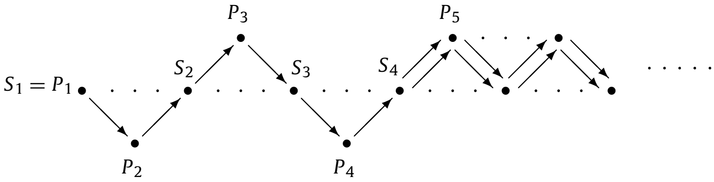
Proof. This follows at once from 4.1 and 1.7. $\square$

4.6. The following example illustrates the theorem.

Example. Let A be given by the quiver

$$\underset{1}{\bullet} \longleftarrow \underset{2}{\bullet} \longleftarrow \underset{3}{\bullet} \longleftarrow \underset{4}{\bullet} \leftleftarrows \underset{5}{\bullet}$$

bound by $\mathrm{rad}^2 A = 0$. Then $\Gamma(\mathrm{mod}\, A)$ has a postprojective component of the form



Taking $\mathcal{C}$ to be the full subcategory consisting of the predecessors of P_4, we see that $\mathrm{pgd}(\mathcal{C}) = 2$. Here, $E = P_2 \oplus P_3 \oplus S_3 \oplus P_4$ and $F = P_5$. Clearly, the conditions of the theorem are satisfied, and $T = E \oplus F$ is a tilting A-module. Notice that E is not convex.

5. Convex tilting modules

5.1. In this section, we apply our main result to generalise [10, (2.1)] which characterises tilted algebras as being those algebras having a convex tilting module of projective dimension at most one.

Theorem. *Let T be a tilting* (*or a cotilting*) *A-module. The following conditions are equivalent*:

(a) *T is convex.*
(b) *$T^{\perp}$ is closed under successors.*
(c) *${}^{\perp}T$ is closed under predecessors.*
(d) *T is a slice module and A is a tilted algebra.*

Proof. Assume that T is a tilting module. The proof in the case of T being cotilting is dual.

(a) *implies* (b). Let $X \in T^{\perp}$ be indecomposable and let $X \rightsquigarrow Y$ be a path in $\mathrm{ind}\, A$. We need to show that $Y \in T^{\perp}$. If this is not the case, then there exist $i > 0$ and an indecomposable summand T_0 of T such that $\mathrm{Ext}^i(T_0, Y) \neq 0$. By 2.1, there is then a path $Y \rightsquigarrow T_0$. On the other hand, since $T^{\perp} \subset \mathrm{Gen}\, T$ by 1.2, we infer that $X \in \mathrm{Gen}\, T$, and so, there exists an indecomposable module $T_1 \in \mathrm{add}\, T$ such that $\mathrm{Hom}_A(T_1, X) \neq 0$. This gives a path

$$T_1 \longrightarrow X \rightsquigarrow Y \rightsquigarrow T_0.$$

Hence, by convexity, $Y \in \mathrm{add}\, T \subset T^{\perp}$. This proves (b).

(b) *implies* (d). Assume that $T^{\perp}$ is closed under successors. Since T is a tilting module then, by 1.3, $T^{\perp}$ is covariantly finite and coresolving. By 2.3, $T^{\perp} = \mathrm{Gen}\, L$, where L is the direct sum of all indecomposable Ext-projective modules in $T^{\perp}$. By 1.5, this implies that $\mathrm{add}\, L = \mathrm{add}\, T$, and hence $T^{\perp} = \mathrm{Gen}\, T$. By [3], there exists a tilting module U such that $\mathrm{pd}\, U \leqslant 1$ and $\mathrm{Gen}\, T = \mathrm{Gen}\, U$. Moreover, U is Ext-projective in $\mathrm{Gen}\, T$, so $\mathrm{add}\, U \subset \mathrm{add}\, T$. Looking at the number of isomorphism classes of indecomposable summands gives $\mathrm{add}\, U = \mathrm{add}\, T$. By [10, (2.1)], T is a slice module and A is a tilted algebra.

Since, clearly, (d) implies (a), we have established the equivalence of (a), (b) and (d). Assume now that these equivalent conditions hold. By (d), T is also a cotilting module. By the proof dual to the proof of (a) implies (b) above, we get that ${}^{\perp}T$ is closed under predecessors. Conversely, if ${}^{\perp}T$ is closed under predecessors, then the dual of the proof that (b) implies (d) shows that T is a slice module and (hence) that A is a tilted algebra. □

5.2. As an immediate consequence, we get our third main result.

Corollary. *An algebra is tilted if and only if it has a convex tilting* (*or cotilting*) *module.*

5.3.

Corollary. *Let $\mathcal{C}$ be a full subcategory of* ind A*, closed under predecessors. Assume that* add $\mathcal{C}$ *is contravariantly finite. Then*:

(a) *E is convex if and only if* $\operatorname{pgd}(\mathcal{C}) \leqslant 1$.
(b) *If, moreover,* add $\mathcal{C}$ *is resolving, then* $\operatorname{pgd}\mathcal{C} \leqslant 1$ *if and only if A is tilted having E as a slice module.*

Proof. (a) Since the necessity follows from [4, (5.3)(a)] and [4, (6.3)], we prove the sufficiency. Assume that E is convex. By 4.2, E is a cotilting ${}_{\mathcal{C}}A$-module. Also, being convex in $\operatorname{mod} A$, it is also convex in $\operatorname{mod}_{\mathcal{C}} A$. By 5.1, E is a slice module and ${}_{\mathcal{C}}A$ is a tilted algebra. Moreover, because of [9, (2.1)], we have

$$\mathcal{C} \subset \operatorname{Pred} E \subset \mathcal{L}_{{}_{\mathcal{C}}A} \subset \mathcal{L}_A$$

and the result is proven.

(b) This follows from (a). □

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